

On Lagrangian Theory for Rotating Charge

Coupled to the Maxwell Field

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Abstract

We justify the Hamilton least action principle for the Maxwell-Lorentz equations with Abraham's rotating extended electron. The main novelty in the proof is application of the variational Poincaré equations on the Lie group $SO(3)$. The variational equations allow to derive the corresponding conservation laws from general Nöther theory of invariants.

¹Supported partly by grants DFG 436 RUS 113/929/0-1 and RFBR 10-01-00578-a

² Supported partly by the Alexander von Humboldt Research Award, and by the Austrian Science Fund (FWF): P22198-N13.

1 Introduction

We justify the Hamilton least action principle for the system of Maxwell-Lorentz equations with a rotating charged particle. Our main contribution is the variational derivation of the *Lorentz torque equation*, see equation (1.3) below.

First recall the case of a finite system of material points (q_i, m_i) . The *angular momentum* is defined by

$$M := \sum q_i \wedge p_i := \sum q_i \wedge m_i \dot{q}_i. \quad (1.1)$$

By the second and the third Newton laws this implies

$$\dot{M} = \sum q_i \wedge \dot{p}_i = \sum q_i \wedge F_i = \sum q_i \wedge F_i^{ext} = T, \quad (1.2)$$

where T is called the *external force torque*. Our aim is to derive the similar *torque equation* for a charged rigid body in the Maxwell field:

$$I\dot{\omega} = e \int (x - q) \wedge [E + E^{ext} + (q + \omega \wedge (x - q)) \wedge (B + B^{ext})] \rho(x - q) dx, \quad (1.3)$$

where I is the moment of inertia, ω is the angular velocity, and $\rho(x)$ is a charge distribution, and the right hand side is the torque of the Lorentz force.

Formally the rigid body can be considered as an infinite system of material points. However equation (1.3) cannot be obtained directly from (1.2), since we cannot correctly take into account all the forces of mutual interaction between the different pieces of the rigid body. That is why we look for a different approach to the derivation of (1.3). We show that (1.3) follows from the Hamilton variational least action principle with the standard interaction term $-A_0\rho + \vec{A} \cdot \vec{j}$ in the Lagrangian density.

For the free rigid body ($E, E^{ext}, B, B^{ext} = 0$) equation (1.3) reduces to the Euler's equations which have been obtained from the variational principle first by Poincaré [10], and in [2] for an external force field with an axial symmetry.

Note that in our case the fields E and B are generated by the motion of the charged body, and E^{ext}, B^{ext} are external fields.

Let us comment on previous works. Equation (1.3) is well recognized since the Abraham's works [3, 4]. In [3, Section 11] Abraham computed the Lagrangian as integral of $-A_0\rho + \vec{A} \cdot \vec{j}$ for standing rotating spherically symmetric electron subject to external fields obeying very special symmetry conditions. In this case the Lagrangian depends only on one variable ω , the angular velocity. However, derivation of the torque equation (1.3) from the variational it Hamilton least action principle remained an open question.

Nodvik applied the variational Hamilton principle to the rotating charge in the Euler angles [9]. He deduced the dynamical equation [9, (5.46)] for $\omega(t)$ and have established the corresponding conservation laws. However, the Nodvik equation looks differently from (1.3). Appel and Kiessling [5] suggested a transformation of [9, (5.46)] to (1.3), but they did not give the details, see [5, A.1.4]. The direct derivation of the conservation laws from (1.3) is presented by Kiessling in [8].

We propose an invariant derivation of (1.3) from the Hamilton least action principle relying on Poincaré equations [10, 2] on the Lie group $SO(3)$. We also deduce the corresponding conservation laws by general Lagrangian formalism using the Nöther theory of invariants.

The plan of our article is as follows. In Section 2 we state the Maxwell-Lorentz equations for the rotating charge, and introduce the corresponding Lagrangian functional. Further we deduce the

equations from the Hamilton variational principle relying on the Poincaré theory [10] presented in the Appendix A. In Section 3 we prove the conservation laws. Finally, in Appendix B we identify the commutators of invariant vector fields.

2 Maxwell-Lorentz equations

The Maxwell field consists of the electric field $E(x, t)$ and the magnetic field $B(x, t)$ generated by a motion of a rotating charge. The external fields E^{ext} and B^{ext} are also generated by the corresponding external charges and currents. Let the rotating charge be centered at the position q with the velocity \dot{q} . For simplicity we assume that the mass distribution, $m\rho(x)$, and the charge distribution, $e\rho(x)$, are proportional to each other. Here m is the total mass, e is the total charge, and we use a system of units such that $m = 1$, $e = 1$. The coupling function $\rho(x)$ is a sufficiently smooth radially symmetric function of fast decay as $|x| \rightarrow \infty$,

$$\rho(x) = \rho_r(|x|). \quad (C)$$

2.1 Angular velocity

Let us denote by $\omega(t) \in \mathbb{R}^3$ the angular velocity “in space” (in the terminology of [2]) of the charge. Namely, let us fix a “center” point O of the rigid body. Then the trajectory of each fixed point of the body is described by

$$x(t) = q(t) + R(t)(x(0) - q(0)),$$

where $q(t)$ is the position of O at the time t , and $R(t) \in SO(3)$. Respectively, the velocity reads

$$\dot{x}(t) = \dot{q}(t) + \dot{R}(t)(x(0) - q(0)) = \dot{q}(t) + \dot{R}(t)R^{-1}(t)(x(t) - q(t)) = \dot{q}(t) + \omega(t) \wedge (x(t) - q(t)), \quad (2.1)$$

where $\omega(t) \in \mathbb{R}^3$ corresponds to the skew-symmetric matrix $\dot{R}(t)R^{-1}(t)$ by the rule

$$\dot{R}(t)R^{-1}(t) = \mathcal{J}\omega(t) := \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}. \quad (2.2)$$

We assume that x and q refer to a certain Euclidean coordinate system in \mathbb{R}^3 , and the vector product \wedge is defined in this system by standard formulas. The identification (2.2) of a skew-symmetric matrix and the corresponding angular velocity vector is true in any Euclidean coordinate system of the same orientation as the initial one.

2.2 Dynamical equations

Then the system of Maxwell-Lorentz equations with spin reads, see [11]

$$\dot{E} = \nabla \wedge B - (\dot{q} + \omega \wedge (x - q))\rho(x - q), \quad \dot{B} = -\nabla \wedge E, \quad (2.3)$$

$$\nabla \cdot E(x, t) = \rho(x - q(t)), \quad \nabla \cdot B(x, t) = 0, \quad (2.4)$$

$$\ddot{q} = \int [E + E^{ext} + (\dot{q} + \omega \wedge (x - q)) \wedge (B + B^{ext})]\rho(x - q)dx, \quad (2.5)$$

$$I \dot{\omega} = \int (x - q) \wedge [E + E^{ext} + (\dot{q} + \omega \wedge (x - q)) \wedge (B + B^{ext})] \rho(x - q) dx, \quad (2.6)$$

where I is the moment of inertia defined by

$$I = \frac{2}{3} \int x^2 \rho(x) dx. \quad (2.7)$$

Here the equations (2.3) are Maxwell equations with the corresponding charge density and current, equations (2.4) are constraints. The back reaction of the field onto the particle is given through the Lorentz force equation (2.5), and the Lorentz torque equation (2.6) deals with rotational degrees of freedom.

2.3 Lagrangian functional and variational principle

Our main goal is to deduce equations (2.3)-(2.6) from the Hamilton least action principle. First let us introduce *electromagnetic potentials* $\mathcal{A} = (A_0, A)$, $\mathcal{A}^{ext} = (A_0^{ext}, A^{ext})$:

$$B = \nabla \wedge A, \quad E = -\nabla A_0 - \dot{A}. \quad (2.8)$$

$$B^{ext} = \nabla \wedge A^{ext}, \quad E^{ext} = -\nabla A_0^{ext} - \dot{A}^{ext}. \quad (2.9)$$

Next we define the Lagrangian

$$\begin{aligned} L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) &= \frac{1}{2} \int (E^2 - B^2) dx + \frac{1}{2} \dot{q}^2 + \frac{1}{2} I \omega^2 - \int [A_0 + A_0^{ext}] \rho(x - q) dx + \\ &\quad \int (\dot{q} + \omega \wedge (x - q)) \cdot [A + A^{ext}] \rho(x - q) dx, \end{aligned} \quad (2.10)$$

where E, B are expressed in terms of $\mathcal{A}, \dot{\mathcal{A}}$ by (2.8), and $\omega = \mathcal{J}^{-1} \dot{R} R^{-1}$ by (2.2).

The last two integrals represent the interaction term

$$\int [(A_0 + A_0^{ext}) \rho - j[A + A^{ext}]] dx$$

in view of (2.1). The corresponding action functional has the form

$$S = S(\mathcal{A}, q, R) := \int_{t_1}^{t_2} L(\mathcal{A}(t), q(t), R(t), \dot{\mathcal{A}}(t), \dot{q}(t), \dot{R}(t)) dt \quad (2.11)$$

Then the Hamilton least action principle reads

$$\delta S(\mathcal{A}, q, R) = 0, \quad (2.12)$$

where the variation is taken over $\mathcal{A}(t), q(t), R(t)$ with the boundary conditions

$$(\delta \mathcal{A}, \delta q, \delta R)|_{t=t_1} = (\delta \mathcal{A}, \delta q, \delta R)|_{t=t_2} = 0. \quad (2.13)$$

We assume that all the involved functions and fields are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x| \rightarrow \infty$ so that partial integrations below could be possible.

Our main result is the following theorem.

Theorem 2.1 *The Maxwell-Lorentz system with spin (2.3) to (2.6) is equivalent to the least action principle (2.12)–(2.13).*

We will analyze the variations in \mathcal{A} , q , R separately, namely, we prove that

$$\frac{\delta S}{\delta \mathcal{A}} = 0 \quad (a), \quad \frac{\delta S}{\delta q} = 0 \quad (b), \quad \frac{\delta S}{\delta R} = 0 \quad (c) \quad (2.14)$$

is equivalent to (2.3)–(2.6).

2.4 Equations for fields and particle trajectory

Equations for fields It is well-known [6] that (2.14) (a) is equivalent to

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\mathcal{A}}} = L_{\mathcal{A}} \quad (2.15)$$

and that the last Euler-Lagrange equations are equivalent to the Maxwell equations (2.3) with the constraints (2.4).

Remark Note that the terms with A_0^{ext} and A^{ext} in (2.10) are additive and remain additive while one makes variations in q , and R . Then, for simplicity of exposition, we put in all of the computations below $A_0^{ext} = 0$ and $A^{ext} = 0$.

Equations for particle trajectory Similarly, (2.14) (b) is equivalent to

$$\frac{d}{dt} L_{\dot{q}} = L_q. \quad (2.16)$$

It remains to check that the last Euler-Lagrange equations are equivalent to the Lorentz force equation (2.5). Let us change variables in the last two integrals of (2.10) and obtain

$$- \int A_0(x+q, t) \rho(x) + \int (\dot{q} + \omega \wedge x) \cdot A(x+q, t) \rho(x) dx.$$

We have

$$L_{\dot{q}} = \dot{q} + \int A(x+q, t) \rho(x) dx, \quad L_q = - \int \nabla A_0(x+q, t) \rho(x) dx + \int (\dot{q} + \omega \wedge x) \cdot \nabla A(x+q, t) \rho(x) dx.$$

Here, for a vector field $b(x)$ we denote $b \cdot \nabla A = \sum b_j \nabla A_j$. Then the Euler-Lagrange equation (2.16) reads

$$\ddot{q} + \int \left(\dot{A}(x+q, t) + (\dot{q} \cdot \nabla) A(x+q, t) \right) \rho dx = - \int \nabla A_0(x+q, t) \rho(x) dx + \int (\dot{q} + \omega \wedge x) \cdot \nabla A(x+q, t) \rho dx.$$

Substituting $\dot{A}(x+q, t) = -E(x+q, t) - \nabla A_0(x+q, t)$ we obtain

$$\ddot{q} = \int E(x+q, t) \rho(x) dx + \int (\dot{q} \cdot \nabla A(x+q, t) - (\dot{q} \cdot \nabla) A(x+q, t)) \rho dx + \int (\omega \wedge x) \cdot \nabla A(x+q, t) \rho dx. \quad (2.17)$$

To make the notations shorter let us omit the dependence on t in the further computations. By the identity $\dot{q} \cdot \nabla A(x+q) - (\dot{q} \cdot \nabla)A(x+q) = \dot{q} \wedge \nabla \wedge A(x+q)$ we obtain

$$\int (\dot{q} \cdot \nabla A(x+q) - (\dot{q} \cdot \nabla)A(x+q))\rho dx = \int \dot{q} \wedge \nabla \wedge A(x+q)\rho dx. \quad (2.18)$$

It remains to check that

$$\int (\omega \wedge x) \cdot \nabla A(x+q)\rho dx = \int \omega \wedge x \wedge \nabla \wedge A(x+q)\rho dx. \quad (2.19)$$

Let us check for the first component, for the rest the computation is similar. The first component of the LHS of (2.19) equals

$$\int [(\omega_2 x_3 - \omega_3 x_2)\partial_1 A_1(x+q) + (\omega_3 x_1 - \omega_1 x_3)\partial_1 A_2(x+q) + (\omega_1 x_2 - \omega_2 x_1)\partial_1 A_3(x+q)]\rho dx.$$

The first component of the RHS of (2.19) equals

$$\int [(\omega_3 x_1 - \omega_1 x_3)(\partial_1 A_2(x+q) - \partial_2 A_1(x+q)) - (\omega_1 x_2 - \omega_2 x_1)(\partial_3 A_1(x+q) - \partial_1 A_3(x+q))]\rho dx.$$

For the difference of the LHS and the RHS we apply partial integration, and obtain

$$\begin{aligned} & \int [(\omega_2 x_3 - \omega_3 x_2)\partial_1 A_1(x+q) + (\omega_3 x_1 - \omega_1 x_3)\partial_2 A_1(x+q) + (\omega_1 x_2 - \omega_2 x_1)\partial_3 A_1(x+q)]\rho dx = \\ & - \int A_1(x+q) [(\omega_2 x_3 - \omega_3 x_2)\partial_1 + (\omega_3 x_1 - \omega_1 x_3)\partial_2 + (\omega_1 x_2 - \omega_2 x_1)\partial_3]\rho dx = \\ & - \int A_1(x+q) [\omega_1(x_2\partial_3 - x_3\partial_2) + \omega_2(x_3\partial_1 - x_1\partial_3) + \omega_3(x_1\partial_2 - x_2\partial_1)]\rho dx = \\ & - \int A_1(x+q)(\omega \cdot \nabla_\theta)\rho dx, \end{aligned}$$

where $\nabla_\theta = (\nabla_{\theta_1}, \nabla_{\theta_2}, \nabla_{\theta_3})$ and ∇_{θ_j} is the differentiation in the angular coordinate θ_j around the coordinate axis x_j . Since ρ is radially symmetric, one has $\nabla_\theta \rho = 0$ and we come to (2.19). From (2.8), (2.18), and (2.19) we conclude that the equation (2.17) reads (2.5).

2.5 The torque equation

Finally, it remains to check that (2.14) (c) is equivalent to (2.6).

To make the corresponding variation in R , let us express $\omega(t) = J^{-1}\dot{R}(t)R^{-1}(t)$ in the right-invariant vector fields on $SO(3)$. Namely, consider an orthonormal basis $\{e_k\}$ with the right orientation in \mathbb{R}^3 . Then

$$e_1 \wedge e_2 = e_3, \quad e_3 \wedge e_1 = e_2, \quad e_2 \wedge e_3 = e_1. \quad (2.20)$$

Let us express the angular velocity in $\{e_k\}$: $\omega(t) = \sum \omega_k(t)e_k$. The space $so(3)$ of skew-symmetric 3×3 matrices with the matrix commutator is isomorphic to \mathbb{R}^3 with vector product by the isomorphism \mathcal{J} of (2.2):

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \mathcal{J}(\omega_1, \omega_2, \omega_3). \quad (2.21)$$

In detail, if $A, B \in so(3)$, $a, b \in \mathbb{R}^3$, and $A = \mathcal{J}a$, $B = \mathcal{J}b$ by the isomorphism (2.21), then

$$AB - BA = \mathcal{J}(a \wedge b). \quad (2.22)$$

Further, $\dot{R}(t)R^{-1}(t) \in T_E SO(3)$ is the tangent vector $\dot{R}(t)$ of $SO(3)$ at the point $R(t)$ translated to the unit E of $SO(3)$ by the right translation $R^{-1}(t)$. By the linear isomorphism (2.21),

$$\dot{R}(t)R^{-1}(t) = \sum \omega_k(t)\tilde{e}_k, \quad \tilde{e}_k := \mathcal{J}^{-1}e_k. \quad (2.23)$$

Then

$$\dot{R}(t) = \dot{R}(t)R^{-1}(t)R(t) = \sum \omega_k(t)v_k(R(t)), \quad v_k(R) := \tilde{e}_k R. \quad (2.24)$$

As the result, $\dot{R}(t)$ has the same coordinates w.r.t. the vector fields v_k at the point $R(t)$ as $\omega(t)$ in the basis $\{e_k\}$. The fields $v_k(R)$ are right translations of \tilde{e}_k and hence are right-invariant.

The next lemma is proved in Appendix B.

Lemma 2.2 *For the constructed above vector fields v_k on $SO(3)$ the following commutation relations hold:*

$$[v_1, v_2] = -v_3, \quad [v_3, v_1] = -v_2, \quad [v_2, v_3] = -v_1. \quad (2.25)$$

Poincaré equations Now we are going to deduce (2.6) from the *Poincaré equations*. Namely, as shown by Poincaré [10, 2], the equation (2.14) (c) is equivalent to the Poincaré equations

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_k} = \sum_{ij} c_{ik}^j \omega_i \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}), \quad k = 1, 2, 3, \quad (2.26)$$

where \hat{L} is L with $\omega(t)$ expressed in the coordinates $(\omega_1(t), \omega_2(t), \omega_3(t))$. For convenience of the reader the derivation of Poincaré equations is presented in Appendix A.

Note that the Lagrangian \hat{L} does not depend explicitly on R , and hence $v_k(\hat{L}) = 0$, $k = 1, 2, 3$ by (A.4). Hence, the corresponding Poincaré equation reads

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_1} = \sum_{ij} c_{i1}^j \omega_i \frac{\partial \hat{L}}{\partial \omega_j}.$$

Equivalence to the torque equation It remains to check that the equations (2.26) are equivalent to the Lorentz torque equation (2.6). It suffices to check the equivalence for the first component with $k = 1$, since for the rest k the computation is similar.

First,

$$\frac{\partial \hat{L}}{\partial \omega_1} = I\omega_1 + \int (x_2 A_3(x+q) - x_3 A_2(x+q)) \rho(x) dx.$$

Therefore,

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_1} = \int (x_2 (\dot{A}_3(x+q) + \dot{q} \cdot \nabla A_3(x+q)) - x_3 (\dot{A}_2(x+q) + \dot{q} \cdot \nabla A_2(x+q))) \rho dx.$$

Second, by (2.25) and (A.5) one has $c_{31}^2 = -1$, $c_{21}^3 = 1$, and all the rest $c_{i1}^j = 0$. Hence,

$$\sum_{ij} c_{i1}^j \omega_i \frac{\partial \hat{L}}{\partial \omega_j} = \omega_2 \frac{\partial \hat{L}}{\partial \omega_3} - \omega_3 \frac{\partial \hat{L}}{\partial \omega_2} =$$

$$\omega_2 \left[I\omega_3 + \int (x_1 A_2(x+q) - x_2 A_1(x+q)) \rho dx \right] - \omega_3 \left[I\omega_3 + \int (x_3 A_1(x+q) - x_1 A_3(x+q)) \rho dx \right] =$$

$$\int [-(\omega_2 x_2 + \omega_3 x_3) A_1(x+q) + x_1(\omega_2 A_2(x+q) + \omega_3 A_3(x+q))] \rho dx.$$

Finally, we come to the equation

$$I\dot{\omega}_1 = \int (x_3 \dot{A}_2(x+q) - x_2 \dot{A}_3(x+q)) \rho dx + \int [x_3(\dot{q} \cdot \nabla) A_2(x+q) - x_2(\dot{q} \cdot \nabla) A_3(x+q)] \rho dx +$$

$$\int [x_1(\omega_2 A_2(x+q) + \omega_3 A_3(x+q)) - (\omega_2 x_2 + \omega_3 x_3) A_1(x+q)] \rho dx. \quad (2.27)$$

Now let us proceed to the first component of the equation (2.6). Using the identity

$$x \wedge [(\omega \wedge x) \wedge B] = (\omega \wedge x)(x \cdot B)$$

we obtain

$$I\dot{\omega}_1 = \int (x \wedge E(x+q))_1 \rho dx + \int (x \wedge (\dot{q} \wedge B(x+q)))_1 \rho dx + \int (\omega \wedge x)_1 (x \cdot B(x+q)) \rho dx, \quad (2.28)$$

We insert $B = \nabla \wedge A$ and obtain that the first integral equals

$$\int [x_3 \dot{A}_2(x+q) - x_2 \dot{A}_3(x+q) + (x_3 \partial_2 - x_2 \partial_3) \varphi] \rho dx =$$

$$\int (x_3 \dot{A}_2(x+q) - x_2 \dot{A}_3(x+q)) \rho dx - \int \varphi (x_3 \partial_2 - x_2 \partial_3) \rho dx =$$

$$\int (x_3 \dot{A}_2(x+q) - x_2 \dot{A}_3(x+q)) \rho dx, \quad (2.29)$$

since $(x_3 \partial_2 - x_2 \partial_3) \rho = \partial_{\theta_1} \rho = 0$. Further, the second integral of (2.28) reads

$$\int [x_2(\dot{q}_1(\partial_3 A_1(x+q) - \partial_1 A_3(x+q)) - \dot{q}_2(\partial_2 A_3(x+q) - \partial_3 A_2(x+q))) -$$

$$x_3(\dot{q}_3(\partial_2 A_3(x+q) - \partial_3 A_2(x+q)) - \dot{q}_1(\partial_1 A_2(x+q) - \partial_2 A_1(x+q)))] \rho dx. \quad (2.30)$$

Finally, the third integral of (2.28) gives

$$\int [x_1(\omega_2 A_2(x+q) + \omega_3 A_3(x+q)) - (\omega_2 x_2 + \omega_3 x_3) A_1(x+q)] \rho dx +$$

$$\int (\omega_2 x_3 - \omega_3 x_2)(A_1(x+q)(x_3 \partial_2 - x_2 \partial_3) + A_2(x+q)(x_1 \partial_3 - x_3 \partial_1) + A_3(x+q)(x_2 \partial_1 - x_1 \partial_2)) \rho dx =$$

$$\int [x_1(\omega_2 A_2(x+q) + \omega_3 A_3(x+q)) - (\omega_2 x_2 + \omega_3 x_3) A_1(x+q)] \rho dx, \quad (2.31)$$

since

$$(A_1(x+q)(x_3 \partial_2 - x_2 \partial_3) + A_2(x+q)(x_1 \partial_3 - x_3 \partial_1) + A_3(x+q)(x_2 \partial_1 - x_1 \partial_2)) \rho = A(x+q) \cdot \nabla_\theta \rho = 0.$$

By (2.29), (2.30), (2.31), the difference between the RHS of (2.27) and the RHS of (2.28) equals

$$\begin{aligned} & \int [x_2(\dot{q}_1 \partial_3 A_1(x+q) + \dot{q}_2 \partial_3 A_2(x+q)) - x_3(\dot{q}_3 \partial_2 A_3(x+q) + \dot{q}_1 \partial_2 A_1(x+q))] \rho dx + \\ & \int [x_2 \dot{q}_3 \partial_3 A_3(x+q) - x_3 \dot{q}_2 \partial_2 A_2(x+q)] \rho dx = \\ & \int \dot{q} \cdot A(x+q)(x_3 \partial_2 - x_2 \partial_3) \rho dx = 0 \end{aligned}$$

and we obtain that the equation (2.27) reads (2.28). The theorem is proved. \square

3 Conservation laws

We have derived the system (2.3)-(2.6) by the least action principle with the Lagrangian (2.10). When the external fields possess a symmetry, the corresponding conservation laws can be also derived from the Lagrangian formalism. Let us recall that the Lagrangian (2.10) reads

$$\begin{aligned} L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) &= \frac{1}{2} \int (E^2 - B^2) dx + \frac{1}{2} \dot{q}^2 + \frac{1}{2} I \omega^2 - \\ & \int (A_0 + A_0^{ext}) \rho(x-q) dx + \int (\dot{q} + \omega \wedge (x-q)) \cdot (A + A^{ext}) \rho(x-q) dx, \end{aligned} \quad (3.1)$$

where $\omega = \mathcal{J}^{-1} \dot{R} R^{-1}$, $\mathcal{A} = (A_0, A)$, $\dot{\mathcal{A}} = (\dot{A}_0, \dot{A})$, and

$$E = -\nabla A_0 - \dot{A}, \quad B = \nabla \wedge A. \quad (3.2)$$

As above, we denote

$$\hat{L}(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) = L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) \quad (3.3)$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is defined by (2.24) i.e. ω_k are coordinates of \dot{R} in the basis $v_1(R), v_2(R), v_3(R)$.

3.1 Energy conservation

Let us denote

$$X := (\mathcal{A}, q, R), \quad V := \dot{X} = (\dot{\mathcal{A}}, \dot{q}, \dot{R}). \quad (3.4)$$

Let A_0^{ext} and A^{ext} do not depend on time. Then the Lagrangian (3.1) does not depend on t , and the *energy*

$$E(X, V) := L_V \cdot V - L \quad (3.5)$$

is conserved, [1]. By (3.4) and since L does not depend on \dot{A}_0 , we have

$$L_V \cdot V = L_{\dot{A}} \cdot \dot{A} + L_{\dot{q}} \cdot \dot{q} + L_{\dot{R}} \cdot \dot{R} = \hat{L}_{\dot{A}} \cdot \dot{A} + \hat{L}_{\dot{q}} \cdot \dot{q} + \hat{L}_\omega \cdot \omega. \quad (3.6)$$

Proposition 3.1 *The energy reads*

$$E = \frac{1}{2} \int (|E|^2 + |B|^2) dx + \frac{1}{2} \dot{q}^2 + \frac{1}{2} I \omega^2 + \int A_0^{ext} \rho(x - q) dx. \quad (3.7)$$

Proof By (3.3) and (3.1), one has

$$\hat{L}_{\dot{A}} \cdot \dot{A} = - \int E \cdot \dot{A} dx, \quad \hat{L}_{\dot{q}} \cdot \dot{q} = \dot{q}^2 + \int \dot{q} \cdot (A + A^{ext}) \rho(x - q) dx,$$

and

$$\hat{L}_{\omega} \cdot \omega = I \omega^2 + \int \omega \wedge (x - q) \cdot (A + A^{ext}) \rho(x - q) dx.$$

Then

$$\begin{aligned} E &= \hat{L}_{\dot{A}} \cdot \dot{A} + \hat{L}_{\dot{q}} \cdot \dot{q} + \hat{L}_{\omega} \cdot \omega - \hat{L} \\ &= \frac{1}{2} \dot{q}^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} \int (|B|^2 - |E|^2) dx + \int (-E \cdot \dot{A} + A_0 \cdot (\nabla \cdot E) + A_0^{ext} \rho(x - q)) dx. \end{aligned} \quad (3.8)$$

The last integral equals

$$\begin{aligned} - \int (E \cdot \dot{A} + \nabla A_0 \cdot E + A_0^{ext} \rho(x - q)) dx &= - \int (E(\dot{A} + \nabla A_0) + A_0^{ext} \rho(x - q)) dx \\ &= \int E^2 dx + \int A_0^{ext} \rho(x - q) dx \end{aligned}$$

and hence (3.8) reads (3.7). \square

3.2 Momentum conservation

Let us consider the spatial translations of the lagrangian coordinate $X = (\mathcal{A}, q, R)$:

$$(\mathcal{A}(x), q, R) \mapsto (\mathcal{A}(x - h), q + h, R).$$

If the external field $\mathcal{A}^{ext} = (A_0^{ext}, A^{ext})$ does not depend on x_j with some j , then the Lagrangian (3.1) is invariant w.r.t to the one-parametric group of spatial translations

$$g_s^j(\mathcal{A}(x), q, R) = (\mathcal{A}(x - s e_j), q + s e_j, R), \quad (3.9)$$

where $e_j \in \mathbb{R}^3$ is the corresponding basis vector. By the Nöther theorem [1] the expression

$$P_j = P_j(X, V) := L_V \cdot \frac{dg_s^j(X)}{ds} \Big|_{s=0} \quad (3.10)$$

is conserved.

Definition 3.2 *Vector $P = (P_1, P_2, P_3)$ is called momentum of the state (X, V) .*

Proposition 3.3 *The momentum reads (cf. [8])*

$$P = \dot{q} + \int E \wedge B dx + \int A^{ext} \rho(x - q) dx. \quad (3.11)$$

Proof For concreteness let us compute P_1 . Formula (3.9) implies

$$\frac{dg_s^1(X)}{ds}|_{s=0} = -(e_1 \cdot \nabla A(x), e_1, 0).$$

Since L does not depend on \dot{A}_0 , and the map g_s^1 leaves ω unchanged,

$$\begin{aligned} P_1 &= L_V \cdot \frac{dg_s^1(X)}{ds}|_{s=0} = -L_{\dot{A}} \cdot (e_1 \cdot \nabla) A + L_{\dot{q}} \cdot e_1 = \\ &= - \int (\nabla A_0 + \dot{A}) \cdot (e_1 \cdot \nabla) A dx + \dot{q} e_1 + \int e_1 \cdot A \rho(x - q) dx + \int A_1^{ext} \rho(x - q) dx. \end{aligned}$$

Note that

$$\begin{aligned} &\dot{q}_1 + \int A_1 \rho(x - q) dx - \int (\nabla A_0 + \dot{A}) \cdot \partial_1 A dx = \\ &\dot{q}_1 + \int A_1 \rho(x - q) dx - \int (\partial_1 A_0 \partial_1 A_1 + \partial_2 A_0 \partial_1 A_2 + \partial_3 A_0 \partial_1 A_3 + \dot{A}_1 \partial_1 A_1 + \dot{A}_2 \partial_1 A_2 + \dot{A}_3 \partial_1 A_3) dx. \end{aligned} \quad (3.12)$$

On the other hand, consider the RHS of (3.11) and obtain

$$\begin{aligned} &\dot{q}_1 + \int (E \wedge B)_1 dx = \dot{q}_1 + \int [(-\partial_2 A_0 - \dot{A}_2)(\partial_1 A_2 - \partial_2 A_1) + (\partial_3 A_0 + \dot{A}_3)(\partial_3 A_1 - \partial_1 A_3)] dx = \\ &\dot{q}_1 + \int (-\partial_2 A_0 \partial_1 A_2 - \dot{A}_2 \partial_1 A_2 + \partial_2 A_0 \partial_2 A_1 + \dot{A}_2 \partial_1 A_2 + \partial_3 A_0 \partial_3 A_1 + \dot{A}_3 \partial_3 A_1 - \partial_3 A_0 \partial_1 A_3 - \dot{A}_3 \partial_1 A_3) dx. \end{aligned} \quad (3.13)$$

The difference between (3.12) and (3.13) equals

$$\int A_1 \rho(x - q) dx - \int (\partial_1 A_0 \partial_1 A_1 + \partial_2 A_0 \partial_2 A_1 + \partial_3 A_0 \partial_3 A_1 + \dot{A}_1 \partial_1 A_1 + \dot{A}_2 \partial_2 A_1 + \dot{A}_3 \partial_3 A_1) dx. \quad (3.14)$$

Finally, the partial integration implies

$$\begin{aligned} \int A_1 \rho(x - q) dx &= \int A_1 (\nabla \cdot E) dx = \int A_1 \nabla \cdot (-\nabla A_0 - \dot{A}) dx = \int A_1 (-\Delta A_0 - \nabla \dot{A}) dx = \\ &= \int (\partial_1 A_0 \partial_1 A_1 + \partial_2 A_0 \partial_2 A_1 + \partial_3 A_0 \partial_3 A_1 + \dot{A}_1 \partial_1 A_1 + \dot{A}_2 \partial_2 A_1 + \dot{A}_3 \partial_3 A_1) dx \end{aligned}$$

Hence, the difference (3.14) equals zero and the first components of the LHS and RHS of (3.11) are equal. \square

3.3 Angular momentum conservation

Let the external potential \mathcal{A}^{ext} be axially symmetric,

$$\mathcal{A}^{ext}(U_k x) = U_k \mathcal{A}^{ext}(x), \quad (3.15)$$

where U_k is any rotation around the axis x_k .

Lemma 3.4 *Let (3.15) hold. Then the Lagrangian (3.1) is invariant w.r.t. the axial rotations*

$$A_0(x) \mapsto A_0(U_k^{-1}x), \quad A(x) \mapsto U_k A(U_k^{-1}x), \quad \dot{A}(x) \mapsto U_k \dot{A}(U_k^{-1}x); \quad (3.16)$$

$$q \mapsto U_k q, \quad \dot{q} \mapsto U_k \dot{q}, \quad (3.17)$$

$$R \mapsto U_k R, \quad \dot{R} \mapsto U_k \dot{R}, \quad (3.18)$$

Proof By (3.2) the transforms (3.16) of the potentials induce the following transforms of the fields:

$$E(x) \mapsto U_k E(U_k^{-1}x), \quad B(x) \mapsto U_k B(U_k^{-1}x). \quad (3.19)$$

Further, we have, in operator notations, $\mathcal{J}\omega = \omega \wedge$, where $\omega \wedge$ is the operator of the vector product by ω in \mathbb{R}^3 . Then it is easy to check that $\mathcal{J}(U_k \omega) = U_k \mathcal{J}(\omega) U_k^{-1}$. Thus, for $\omega = \mathcal{J}^{-1} \dot{R} R^{-1}$ we obtain $\mathcal{J}\omega = \dot{R} R^{-1}$ and hence $\mathcal{J}(U_k \omega) = U_k (\dot{R} R^{-1}) U_k^{-1} = (U_k \dot{R}) (U_k R)^{-1}$. Finally,

$$U_k \omega = \mathcal{J}^{-1}(U_k \dot{R}) (U_k R)^{-1}.$$

This means that the transforms (3.18) induce the following transform of ω :

$$\omega \mapsto U_k \omega. \quad (3.20)$$

Now it is easy to check, in view of axial symmetry of \mathcal{A}^{ext} , the invariance of L w.r.t. the transforms (3.19), (3.17), (3.20) since ρ is spherical symmetric. \square

Recall that \tilde{e}_k is the preimage of the basis vector e_k w.r.t. the isomorphism (2.21). The Lagrangian \hat{L} (3.3) is invariant w.r.t. the spatial rotations (3.19), (3.17), (3.20). In particular, \hat{L} is invariant under the transform groups $g_s^k = e^{s\tilde{e}_k}$. Hence, by the Nöther theorem the expression

$$M_k = M_k(X, V) := \hat{L}_V \cdot \frac{dg_s(X)}{ds} \Big|_{s=0} \quad (3.21)$$

is conserved.

Definition 3.5 *Vector $M = (M_1, M_2, M_3)$ is called angular momentum of the state (X, V) .*

Proposition 3.6 *The angular momentum reads*

$$M = q \wedge \dot{q} + I\omega + \int x \wedge E \wedge B dx + \int x \wedge A^{ext} \rho(x - q) dx. \quad (3.22)$$

Proof For concreteness let us compute M_k with $k = 1$. Then $g_s = e^{s\tilde{e}_1}$, and hence

$$g_s X = (A_0(e^{-s\tilde{e}_1}x), e^{s\tilde{e}_1}A(e^{-s\tilde{e}_1}x), e^{s\tilde{e}_1}q, e^{s\tilde{e}_1}).$$

Then

$$\frac{d}{ds}g_s(X)|_{s=0} =$$

$$(-\tilde{e}_1 e^{-s\tilde{e}_1}x \cdot \nabla)A_0(e^{-s\tilde{e}_1}x), \tilde{e}_1 e^{s\tilde{e}_1}A(e^{-s\tilde{e}_1}x) + e^{s\tilde{e}_1}(-\tilde{e}_1 e^{-s\tilde{e}_1}x \cdot \nabla)A(e^{-s\tilde{e}_1}x), \tilde{e}_1 e^{s\tilde{e}_1}q, \tilde{e}_1 e^{s\tilde{e}_1})|_{s=0}$$

$$= (\tilde{e}_1 A_0(x), \tilde{e}_1 A(x) - (\tilde{e}_1 x \cdot \nabla)A(x), \tilde{e}_1 q, e_1).$$

Note that the last component is the coordinates of \tilde{e}_1 w.r.t. the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and thus equals $e_1 = (1, 0, 0)$. Then, since \hat{L} does not depend on \dot{A}_0 ,

$$\begin{aligned} M_1 &= \hat{L}_V \cdot \frac{d}{ds}g_s(X)|_{s=0} = \hat{L}_{\dot{A}} \cdot (\tilde{e}_1 A(x) - (\tilde{e}_1 x \cdot \nabla)A(x)) + \hat{L}_{\dot{q}} \cdot (\tilde{e}_1 q) + \hat{L}_\omega \cdot e_1 \\ &= \int dx \left(\dot{A} \cdot (\tilde{e}_1 A(x) - (\tilde{e}_1 x \cdot \nabla)A(x)) + \nabla A_0 \cdot (\tilde{e}_1 A(x) - (\tilde{e}_1 x \cdot \nabla)A(x)) \right) \\ &\quad + \dot{q} \cdot (\tilde{e}_1 q) + \int (\tilde{e}_1 q) \cdot (A + A^{ext})\rho(x - q)dx + I\omega \cdot e_1 + \int (e_1 \wedge (x - q)) \cdot [A + A^{ext}]\rho(x - q)dx \\ &= (q \wedge \dot{q})_1 + I\omega_1 + \int (x_2 A_3^{ext} - x_3 A_2^{ext})\rho(x - q)dx \\ &\quad + \int (x_2 A_3 - x_3 A_2)\rho(x - q)dx + \int (\dot{A} + \nabla A_0) \cdot ((0, -A_3, A_2) + (x_3 \partial_2 - x_2 \partial_3)A)dx. \end{aligned} \quad (3.23)$$

We have to prove that this expression equals to the first component of the RHS of (3.22). It suffices to prove that the last line equals to the first component of $\int x \wedge (E \wedge B)dx$. Indeed, $\rho(x - q) = \nabla \cdot E = \nabla \cdot (-\nabla A_0 - \dot{A})$, hence

$$\begin{aligned} \int (x_2 A_3 - x_3 A_2)\rho(x - q)dx &= \int (x_2 A_3 - x_3 A_2)(-\nabla \dot{A} - \nabla^2 A_0)dx \\ &= \int \nabla(x_2 A_3 - x_3 A_2)(\dot{A} + \nabla A_0)dx. \end{aligned} \quad (3.24)$$

Then (3.23) transforms to

$$\begin{aligned} &\int \left(\partial_1(x_2 A_3 - x_3 A_2)(\dot{A}_1 + \partial_1 A_0) + x_2 \partial_2 A_3(\dot{A}_2 + \partial_2 A_0) - x_3 \partial_3 A_2(\dot{A}_3 + \partial_3 A_0) \right) dx \\ &+ \int \left((x_3 \partial_2 - x_2 \partial_3)A_1(\dot{A}_1 + \partial_1 A_0) - x_2 \partial_3 A_2(\dot{A}_2 + \partial_2 A_0) + x_3 \partial_2 A_3(\dot{A}_3 + \partial_3 A_0) \right) dx. \end{aligned} \quad (3.25)$$

On the other hand, substitute $E = -\dot{A} - \nabla A_0$, $B = \nabla \wedge A$ and obtain that the first component of $\int x \wedge (E \wedge B) dx$ equals

$$\begin{aligned} & \int x_2((\partial_1 A_3 - \partial_3 A_1)(\dot{A}_1 + \partial_1 A_0) + (\partial_2 A_3 - \partial_3 A_2)(\dot{A}_2 + \partial_2 A_0)) dx \\ & - \int x_3((\partial_3 A_2 - \partial_2 A_3)(\dot{A}_3 + \partial_3 A_0) + (\partial_1 A_2 - \partial_2 A_1)(\dot{A}_1 + \partial_1 A_0)) dx \end{aligned}$$

which coincides with (3.25). The proof is complete. \square

A Poincaré equations

The derivation of Poincaré equations is presented for the convenience of the reader, our exposition follows [2]. Poincaré has obtained the form of the Hamilton least action principle for Lagrangian systems on manifolds [10].

Let v_1, \dots, v_n be vector fields on a n -dimensional manifold M which are linearly independent at every point. Then the commutation relations hold,

$$[v_i, v_j](g) = \sum c_{ij}^k(g) v_k(g), \quad g \in M$$

where the commutator $[v_i, v_j]$ is defined by

$$[v_i, v_j](f) := v_i(v_j(f)) - v_j(v_i(f)),$$

and $v(f)$ is the derivative of a smooth function f on M w.r.t. the vector field v .

If $g(t)$ is a smooth path in M and f is a smooth function on M , one has $\dot{g}(t) = \sum \omega_i(t) v_i(g(t))$ and

$$\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot \dot{g} = f'(g(t)) \cdot \sum \omega_i(t) v_i(g(t)) = \sum v_i(f) \omega_i(t).$$

Now consider a variation $g(\varepsilon, t)$ of the path $g(t)$. Then similarly,

$$\partial_\varepsilon f(g(\varepsilon, t)) = \sum_j v_j(f) w_j(\varepsilon, t),$$

where $w_j(\varepsilon, t)$ are coordinates of $\frac{\partial g}{\partial \varepsilon}(\varepsilon, t) \in T_{g(\varepsilon, t)} M$. Hence

$$\partial_\varepsilon \partial_t f(g(\varepsilon, t)) = \sum_i \sum_j v_j(v_i(f)) w_j \omega_i + \sum_i v_i(f) \omega'_i,$$

$$\partial_t \partial_\varepsilon f(g(\varepsilon, t)) = \sum_j \sum_i v_i(v_j(f)) w_j \omega_i + \sum_j v_j(f) \dot{w}_j,$$

where the prime resp. dot stand for the differentiation in ε resp. t . However, the differentiations in t and ε commute, hence we obtain by subtraction

$$\sum_k v_k(f) \omega'_k = \sum_k \sum_{ij} c_{ij}^k \omega_i w_j v_k(f) + \sum_k v_k(f) \dot{w}_k.$$

Since f is an arbitrary smooth function, we come to the relations

$$\omega'_k(\varepsilon, t) = \sum_{ij} c_{ij}^k \omega_i w_j + \dot{w}_k. \quad (\text{A.1})$$

Further, let us consider a Lagrangian function $L(\dot{g}, g)$ on TM . Then $L(\dot{g}, g)$ can be expressed in the variables ω : $L(\dot{g}, g) = \hat{L}(\omega, g)$. Let us compute the variation of the corresponding action functional taking (A.1) into account:

$$\begin{aligned} \frac{d}{d\varepsilon} \int_{t_1}^{t_2} \hat{L}(\omega(\varepsilon, t), g(\varepsilon, t)) dt &= \int_{t_1}^{t_2} \left(\sum_k \frac{\partial \hat{L}}{\partial \omega_k} \omega'_k + \nabla_g \hat{L} \cdot g' \right) dt = \\ &= \int_{t_1}^{t_2} \left[\sum_k \frac{\partial \hat{L}}{\partial \omega_k} (\dot{w}_k + \sum_{ij} c_{ij}^k \omega_i w_j) + \nabla_g \hat{L} \cdot \sum_k w_k v_k \right] dt = \\ &= \sum_k \frac{\partial \hat{L}}{\partial \omega_k} w_k \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_k \left[-\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_k} + \sum_{ij} c_{ik}^j \omega_i \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}) \right] w_k dt. \end{aligned}$$

The variation should be zero by the Hamilton least action principle, under the boundary value conditions

$$g(\varepsilon, t_1) = g_1, \quad g(\varepsilon, t_2) = g_2. \quad (\text{A.2})$$

Since $w_k(t_1) = w_k(t_2) = 0$ by (A.2), we obtain the following *Poincaré equations*:

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_k} = \sum_{ij} c_{ik}^j \omega_i \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}). \quad (\text{A.3})$$

Remarks 1. If g is expressed in a local map as $(g_1, \dots, g_n) \in \mathbb{R}^n$, and $v_k = \partial_{g_k}$, then (A.3) reduce to the standard Euler-Lagrange equations.

2. If a Lagrangian L does not depend on g , $\hat{L} = \hat{L}(\omega)$ one has

$$v_k(\hat{L}) = 0. \quad (\text{A.4})$$

Indeed, $v_k(\hat{L}) = \nabla_g \hat{L} \cdot v_k(g) = 0$.

3. Suppose $M = G$ is a Lie group, and let v_k , $k = 1, \dots, n$ be independent either left-invariant or right-invariant vector fields on G . Then $c_{ij}^k(g)$ are constant:

$$c_{ij}^k(g) \equiv c_{ij}^k, \quad g \in G. \quad (\text{A.5})$$

B Commutators of invariant vector fields

Step 1. By (2.22) the isomorphism (2.21) translates relations (2.20) to

$$[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_3, \quad [\tilde{e}_3, \tilde{e}_1] = \tilde{e}_2, \quad [\tilde{e}_2, \tilde{e}_3] = \tilde{e}_1 \quad (\text{B.1})$$

in the sense of matrix commutator.

Step 2. Recall that the right-invariant vector fields v_k on $SO(3)$ are defined by right translations $v_k(R) = \tilde{e}_k R$, where $R \in SO(3)$. We should prove (2.25) in the sense of the commutators of vector fields on the Lie group $SO(3)$.

Since the fields v_k are right-invariant, it suffices to check the relations (2.25) at the group unit E . Let us compute the derivative of a smooth function f on $SO(3)$ w.r.t. a right-invariant field v_A such that $v_A(E) = A \in \mathfrak{so}(3)$. In this case $v_A(R) = AR$ for $R \in SO(3)$. Consider a smooth path $R_1(t) \in SO(3)$ such that $R_1(0) = R$, $\dot{R}_1(0) = AR$. Then

$$v_A(f)(R) = \frac{d}{dt}f(R_1(t))|_{t=0} = \left[f'(R_1(t)) \cdot \dot{R}_1(t) \right] |_{t=0} = f'(R) \cdot AR.$$

In particular,

$$v_{[A,B]}(f)(E) = f'(E) \cdot [A, B], \quad (\text{B.2})$$

where $[A, B] = AB - BA$ is the matrix commutator in $\mathfrak{so}(3)$.

Now let us compute $v_A(v_B(f))(E)$ for a right-invariant field v_B such that $v_B(E) = B \in \mathfrak{so}(3)$, $v_B(R) = BR$. Consider a smooth path $R_2(t) \in SO(3)$ such that $R_2(0) = E$, $\dot{R}_2(0) = A$. Then

$$\begin{aligned} v_A(v_B(f))(E) &= \frac{d}{dt}[f'(R_2(t)) \cdot BR_2(t)]|_{t=0} = \frac{d}{dt}f'(R_2(t))|_{t=0} \cdot BR_2(t)|_{t=0} + f'(R_2(t))|_{t=0} \cdot \frac{d}{dt}BR_2(t)|_{t=0} \\ &= \left[f''(R_2(t)) \cdot \dot{R}_2(t) \right] |_{t=0} \cdot BR_2(t)|_{t=0} + [f'(R(t)) \cdot B\dot{R}_2(t)]|_{t=0} \\ &= (f''(E) \cdot A) \cdot B + f'(E) \cdot BA. \end{aligned}$$

Then, since the form $(f''(E) \cdot A) \cdot B$ is symmetric w.r.t. matrices A, B one has

$$[v_A, v_B](f)(E) = v_A(v_B(f))(E) - v_B(v_A(f))(E) = f'(E) \cdot (BA - AB) = -v_{[A,B]}(f)(E).$$

by (B.2). Together with (B.1) this completes the proof.

References

- [1] V. Arnold, Mathematical methods of classical mechanics, Springer, New York, 1978.
- [2] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt, Mathematical aspects of classical and celestial mechanics, Springer, Berlin, 1997.
- [3] M. Abraham, Prinzipien der Dynamik des Elektrons, Annalen der Physik **10** (1903), 105-179.
- [4] M. Abraham, Theorie der Elektrizität, vol. II, Elektromagnetische Theorie der Strahlung, 2-nd edition, Teubner, Leipzig, 1908.
- [5] W. Appel, M. Kiessling, Mass and spin renormalization in Lorentz electrodynamics, Ann. Phys. **289** (2001), 24-83.
- [6] H. Goldstein, Classical mechanics, 2-nd edition, Addison-Wesley, Reading, MA, 1980.
- [7] J.D. Jackson, Classical electrodynamics, 3-rd edition, N.-Y., Wiley, 1999.

- [8] M. Kiessling, (1999) Classical electron theory and conservation laws, Comm Phys Lett A 258 (1999), 197-204
- [9] J.S. Nodvik, A covariant formulation of classical electrodynamics for charges of finite extensions, Ann. Phys., **28** (1964), 225-319.
- [10] H. Poincaré, Sur une forme nouvelle des équations de la mécanique C. R. **132** (1901), 369-371.
- [11] H. Spohn, Dynamics of charged particles and their radiation field, Cambridge University Press, Cambridge, 2004.